ROBUST BAYESIAN RELIABILITY FOR COMPLEX SYSTEMS UNDER PRIOR-DATA CONFLICT

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Abstract

This paper considers quantification of system reliability in scenarios where data, that is failures or absence of failures, occurring from the system's use over time, are considered to be surprising from the perspective of prior information. A generalized, or imprecise, Bayesian approach is presented for general system structures, where component lifetimes have Weibull distributions with known shape parameter. For the scale parameter, a specific set of prior distributions is assumed which enables prior-data conflict to be reflected through increased imprecision in posterior reliability bounds.

INTRODUCTION

In this paper, we consider a system consisting of components can be divided into K different groups or types, with type k components, for $k = 1, \ldots, K$, assumed to have exchangeable failure times, whilst components of different types are assumed to have independent failure times. We aim to predict the reliability of a running system, assuming we get component failure times from this system up to the moment t_{now} of consideration. We use the Bayesian approach for component failure time distributions, where the ability to incorporate expert knowledge through prior distributions is important because, for most real-world systems, one does not have many component failure times. In line with a generalized setting, advocated both as 'robust Bayesian methods' (Berger et al. 1994) and 'imprecise probability theory' (Walley 1991; Augustin et al. 2014), we use a set of prior distributions which enables a level of indeterminacy to be taken into account on the prior information, which is particularly important in practical scenarios where for example prior knowledge may be mostly based on experience of similar but not identical components, or of such components functioning in different systems or under different circumstances.

While the use of sets of prior distributions in Bayesian statistics has been widely presented in the literature, the key contribution of this paper is the presentation, in the system reliability context, of a specific set of prior distributions for the shape parameter of the Weibull distribution, such that conflict between prior judgements and process data on the components leads to increased imprecision for the system reliability function, hence such conflict is explicitly reported. Prior-data conflict has attracted some attention in the literature (see, e.g., Evans and Moshonov 2006), but less than one may have expected because of the obvious importance of noticing a considerable difference between prior judgements and process data. In standard Bayesian methods, the posterior distribution is always effectively presenting a weighted average of the prior beliefs and the data, without any means to show the level of disagreement between these two sources of information. It is particularly surprising that researchers in robust Bayesian methods, who advocated the use of sets of priors to more adequately represent lack of perfect expert knowledge, did not put more effort in study of prior-data conflict and the possibility to use sets of prior distributions that enable such conflict to be clearly indicated.

The approach used in this paper was introduced by Walter and Augustin 2009 (see also Walter 2013 §3.1.4). It proposes the use of a specific set of prior distributions leading to increased imprecision in posterior inferences when prior knowledge and data are in conflict than when these two sources of information are pretty much in agreement. This paper presents the first use of this statistical method for inference on component failure times, combined to provide predictions of a system's remaining time until failure.

While we advocate the use of a specific set of prior distributions, in a generalized Bayesian framework, for dealing with prior-data conflict, there have also been proposals to reflect such conflict within the standard 'precise' Bayesian approach. For example, Bousquet (2008) proposed a method for diagnostics of agreement or disagreement between the prior and data, based on a ratio of Kullback-Leibler divergences. O'Hagan and Pericchi (2012) propose the use of heavy-tailed distributions which, in case of conflict between multiple sources of prior information, provide more weight to one or more of these, when combined with the data, than would happen with more commonly used prior distributions. Such methods quantify level of disagreement between prior judgements and data, yet they do not resolve the fundamental issue that the end result of the standard Bayesian method remains a single-valued probability which is a weighted average of the prior and data inputs.

The paper is organized as follows. Section 2 introduces the Bayesian model for the Weibull distribution with fixed shape parameter, as used later in this paper for component failure times. Section 3 presents the use of specific sets of priors for the scale parameter of the Weibull distribution, in order to reflect prior-data conflict. Section 4 presents a method to calculate system reliability bounds for a running system based on such sets of priors. An important aspect of the presented method is that it can be used for any system structure, through the use of the 'survival signature'. Section 5 discusses elicitation of sets of prior distributions. Section 6 contains examples illustrating the merits of our method, by studying the effect of surprisingly early or late component failures, showing that observations in conflict to prior assumptions indeed lead to more cautious system reliability predictions. Section 7 concludes the paper by summarizing results and discussing topics for further research.

It should be noted that this paper builds on a paper presented at the 2015 ESREL conference and published in its proceedings (Walter et al. 2015). The next two sections, in which the core generalized Bayesian method for inference about component lifetimes is introduced, follow that paper closely. Thereafter, the material presented in this paper goes far beyond the earlier work: while before only a parallel system consisting of a single type of components was considered, this paper presents methodology for any system structure consisting of multiple types of components, with prior-data conflict occurring at component level yet being reflected at full system level.

BAYESIAN ANALYSIS OF WEIBULL LIFETIMES

In this paper, we consider a system with components of K different types. There are n_k components of type k ($k = 1, \ldots, K$) in the systems, their failure times T_i^k ($i = 1, \ldots, n_k$, $k = 1, \ldots, K$) are assumed to have a Weibull distribution with fixed shape parameter $\beta_k > 0$ and unknown scale parameter λ_k. This distribution has the following probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$,

$$f(t_i^k \mid \lambda_k) = \frac{\beta_k}{\lambda_k} (t_i^k)^{\beta_k - 1} e^{-\frac{(t_i^k)^{\beta_k}}{\lambda_k}}, \tag{1}$$

$$F(t_i^k \mid \lambda_k) = 1 - e^{-\frac{(t_i^k)^{\beta_k}}{\lambda_k}} = P(T_i^k \leq t_i^k \mid \lambda_k), \tag{2}$$

where $\lambda_k > 0$ and $t > 0$.

The shape parameter β_k determines whether the hazard rate is constant ($\beta_k = 1$), increasing ($\beta_k > 1$) or decreasing ($\beta_k < 1$) over time. Throughout this paper we assume that β_k is fixed, this may e.g. be based on engineering knowledge and insights into the practical failure causes. The scale parameter λ_k can be interpreted through the relation

$$\mathrm{E}[T_i^k \mid \lambda_k] = \lambda_k^{1/\beta_k}\, \Gamma(1 + 1/\beta_k)\,. \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function.

In the Bayesian framework, a convenient choice of prior distribution for the scale parameter λ_k is the inverse Gamma distribution, characterized by the probability density function (with hyperparameters $a_k > 0$ and $b_k > 0$)

$$f(\lambda_k \mid a_k, b_k) = \frac{(b_k)^{a_k}}{\Gamma(a_k)} \lambda_k^{-a_k - 1} e^{-\frac{b_k}{\lambda_k}}\,. \tag{4}$$

We indicate this distribution by $\lambda_k \mid a_k, b_k \sim \mathrm{IG}(a_k, b_k)$. The inverse Gamma is convenient because it is a conjugate prior, i.e., the posterior obtained by Bayes' rule is again inverse Gamma and thus easily tractable; the prior parameters only need to be updated to obtain the posterior parameters.

For the imprecise approach presented in this paper, using a set of inverse Gamma prior distributions for λ_k, we use a different parametrization which is more convenient to specify and interpret the set of prior distributions, and hence also to interpret the set of corresponding posterior distributions. We use $n^{(0)} > 1$ and $y^{(0)} > 0$ instead of a_k and b_k, where we drop the index k for the discussion about the prior model in the following, keeping in mind that each component type will have its own specific parameters. Let $n^{(0)} = a - 1$ and $y^{(0)} = b/n^{(0)}$, where $y^{(0)}$ can be interpreted as the prior guess for the scale parameter λ, as $\mathrm{E}[\lambda \mid n^{(0)}, y^{(0)}] = y^{(0)}$. This parametrization also makes the nature of the combination of prior information and data through Bayes' rule very clear: After observing n component lifetimes $\boldsymbol{t} = (t_1, \dots, t_n)$, the updated parameters are

$$n^{(n)} = n^{(0)} + n\,, \qquad\qquad y^{(n)} = \frac{n^{(0)} y^{(0)} + \tau(\boldsymbol{t})}{n^{(0)} + n}\,, \tag{5}$$

where $\tau(\boldsymbol{t}) = \sum_{i=1}^{n} (t_i)^{\beta}$. Hence, the posterior distribution for λ is

$$\lambda \mid n^{(0)}, y^{(0)}, \boldsymbol{t} \sim \mathrm{IG}(n^{(0)} + n + 1, n^{(0)} y^{(0)} + \tau(\boldsymbol{t}))\,. \tag{6}$$

The update rule (5) shows that $y^{(n)}$ is a weighted average of the prior parameter $y^{(0)}$ and the maximum likelihood (ML) estimator $\tau(\boldsymbol{t})/n$, with weights proportional to $n^{(0)}$ and n, respectively. This enables $n^{(0)}$ to be interpreted as a prior strength or pseudocount, indicating how much our prior guess should weigh against the n observations. Furthermore, $\mathrm{Var}[\lambda \mid n^{(0)}, y^{(0)}] = (y^{(0)})^2/(1 - 1/n^{(0)})$, so for fixed $y^{(0)}$, the higher $n^{(0)}$, the more probability mass is concentrated around $y^{(0)}$.

However, the weighted average structure for $y^{(n)}$ is behind the problematic behaviour in case of prior-data conflict; we illustrate this using a small example. Assume an expert judges that a component has a mean lifetime of 9 weeks. Using (3) with $\beta = 2$, we obtain $y^{(0)} = 103.13$. We choose $n^{(0)} = 2$, so our prior guess for the mean component lifetime counts like having two observations with this mean. Suppose that two observations of failures of such a component become available which are surprisingly small, say $t_1 = 1$ and $t_2 = 2$. Using (5) we get $n^{(2)} = 4$ and $y^{(2)} = \frac{1}{4}(2 \cdot 103.13 + 1^2 + 2^2) = 52.82$, so the posterior expectation for the scale parameter λ is 52.82, equivalent to a mean component lifetime of 6.44 weeks. The posterior standard deviation (sd) for λ is 60.99. Compared to the prior standard deviation of 145.85, the posterior expresses now more confidence that mean lifetimes are around $y^{(2)} = 52.82$ than the prior had about $y^{(0)} = 103.13$. This irritating conclusion is illustrated in Figure 1; the posterior cdf is shifted halfway towards the values for λ that the two observations suggest (the ML estimator for λ would be 2.5), and is steeper than the prior (so the pdf is more pointed), thus conveying a false sense of certainty about λ. We would obtain almost the same posterior distribution if we had assumed the mean component lifetime to be 7 weeks (so $y^{(0)} = 62.39$), and observed lifetimes $t_1 = 6$, $t_2 = 7$ in line with our expectations. The main reason for developing the generalized Bayes approach with sets of priors, in order to reflect prior-data conflict, is that it seems unreasonable to make the same probability statements on component lifetimes in these two fundamentally different scenarios.

Note that this is a general problem in Bayesian analysis with canonical conjugate priors. For such priors, the same update formula (5) applies, and so conflict is averaged out, for details see Walter and Augustin (2009) and Walter (2013, §3.1.4 and §A.1.2).

MODELS REFLECTING SURPRISING DATA

Despite the above mentioned issue of ignoring prior-data conflict, the tractability of the update step is a very attractive feature of the conjugate setting used above. As was shown by Walter and Augustin (2009) (see also Walter 2013, §3.1), it is possible to retain tractability and to have a meaningful reaction to prior-data conflict when using sets of priors generated by varying both $n^{(0)}$ and $y^{(0)}$. Then, the magnitude of the set of posteriors, and with it the precision of posterior probability statements, will be sensitive to the degree of prior-data conflict, leading to more cautious probability statements when prior-data conflict occurs.

The generalized Bayes approach for the model for a component's lifetime in this paper is as follows. Instead of a single prior guess $y^{(0)}$ for the mean component lifetimes, a range of prior guesses $[\underline{y}^{(0)}, \overline{y}^{(0)}]$ is used, together with a range $[\underline{n}^{(0)}, \overline{n}^{(0)}]$ of pseudocounts. The set of prior distributions considered is therefore

$$\mathcal{M}^{(0)} := \{f(\lambda \mid n^{(0)}, y^{(0)}) \mid n^{(0)} \in [\underline{n}^{(0)}, \overline{n}^{(0)}], y^{(0)} \in [\underline{y}^{(0)}, \overline{y}^{(0)}]\} \tag{7}$$

Each of the priors $f(\lambda \mid n^{(0)}, y^{(0)})$ is then updated to the posterior $f(\lambda \mid n^{(0)}, y^{(0)}, \boldsymbol{t}) = f(\lambda \mid n^{(n)}, y^{(n)})$ by using (5), such that the set of posteriors $\mathcal{M}^{(n)}$ becomes $\mathcal{M}^{(n)} = \{f(\lambda \mid n^{(n)}, y^{(n)}) \mid n^{(0)} \in [\underline{n}^{(0)}, \overline{n}^{(0)}], y^{(0)} \in [\underline{y}^{(0)}, \overline{y}^{(0)}]\}$. This procedure of using Bayes' Rule element by element is seen as self-evident in the robust Bayesian literature, but can be formally justified as being *coherent* (a self-consistency property) in the framework of imprecise probability, where it is known as *Generalized Bayes' Rule* (Walley 1991, §6.4).

In order to reflect possible prior-data conflict, it is crucial to consider a range of pseudocounts $[\underline{n}^{(0)}, \overline{n}^{(0)}]$ along with the range of prior guesses $[\underline{y}^{(0)}, \overline{y}^{(0)}]$, as only in this case $\tau(\boldsymbol{t})/n \notin [\underline{y}^{(0)}, \overline{y}^{(0)}]$ leads to the set of posteriors being larger and hence reflecting prior-data conflict, as will be illustrated below and in the subsequent sections.

Continuing the example from Section 2 and Figure 1, assume now for the mean component lifetimes the range 9 to 11 weeks, this corresponds to $[\underline{y}^{(0)}, \overline{y}^{(0)}] = [103.13, 154.06]$. Choosing $[\underline{n}^{(0)}, \overline{n}^{(0)}] = [2, 5]$ means that this information on the mean component lifetimes is considered to be of equivalent value to having to two to five actual observations. Compare now the set of posteriors obtained from observing $t_1 = 1$, $t_2 = 2$ (as before), see Figure 2 (left), and $t_1 = 10$, $t_2 = 11$, see Figure 2 (right). There is now a clear difference between the two scenarios of observations in line with expectations and observations in conflict. In the prior-data conflict case, the set of posteriors (blue) is shifted towards the left, but has about the same size as the set of priors (yellow), and so posterior quantification of reliability has the same precision, despite having seen two observations. Instead, in the no conflict case, the set of posteriors is smaller than the set of priors, such that the two observations have increased the precision of reliability statements.

As each posterior in $\mathcal{M}^{(n)}$ corresponds to a predictive distribution for T_{sys}, we will have a set of reliability functions $R_{\text{sys}}(t)$. The derivation of $R_{\text{sys}}(t)$ for systems with K component types and arbitrary layout will be given in Section 4 below. This will include, in contrast to previous studies using sets of priors of type (7), the treatment of censored observations. More specifically, we consider the case of *non-informative right censoring*, where the censoring process is independent of the failure process.

ROBUST RELIABILITY FOR COMPLEX SYSTEMS VIA THE SURVIVAL SIGNATURE

Consider now a system of arbitrary layout, consisting of components of K types. This system is observed until time t_{now}, leading to censored observation of lifetimes of components within the system. We will first explain how the scale parameter λ_k for component type k can be estimated in this situation; then we describe how the system reliability function can be efficiently obtained using the survival signature. To use this approach, we need to derive the posterior predictive distribution of the number of components that function at times $t > t_{\text{now}}$. Finally, we describe how the lower and upper bound for the system reliability function are obtained when prior parameters $(n_k^{(0)}, y_k^{(0)})$ vary in sets $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$, defining sets $\mathcal{M}_k^{(0)}$ of prior distributions over λ_k, as introduced above.

Bayesian Estimation of Component Scale Parameter with Right-censored Lifetimes

Consider observing a system until t_{now}, where the system has K different types of components, and for each type k there are n_k components in the system. Denoting the number of type k components that have failed by t_{now} by e_k, there are $n_k - e_k$ components still functioning at t_{now}. We denote the corresponding vector of observations by

$$\boldsymbol{t}^k_{e_k;n_k} = (\ \underbrace{t_1^k, \dots, t_{e_k}^k}_{e_k \text{failure times}},\ \underbrace{t^+_{\text{now}}, \dots, t^+_{\text{now}}}_{n_k - e_k \text{censored obs.}}\ ), \tag{8}$$

where t^+ indicates a right-censored observation.

According to Bayes' rule, multiplying the prior density and the likelihood (which accounts for right-censored observations through the cdf terms) gives a term proportional to the density of the posterior distribution for λ_k:

$$f(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) \propto f(\lambda_k)[1 - F(t_{\text{now}} \mid \lambda_k)]^{n_k - e_k} \prod_{i=1}^{e_k} f(t_i^k \mid \lambda_k) \tag{9}$$

Conjugacy is preserved and we get $\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k} \sim \text{IG}(n_k^{(n)} + 1, n_k^{(n)} y_k^{(n)})$, where

$$n_k^{(n)} + 1 = n_k^{(0)} + e_k + 1 \tag{10}$$

$$n_k^{(n)} y_k^{(n)} = n_k^{(0)} y_k^{(0)} + (n_k - e_k)(t_{\text{now}})^\beta + \sum_{i=1}^{e_k} (t_i^k)^\beta \tag{11}$$

are the updated parameters of the inverse gamma distribution.

System Reliability using the Survival Signature

The structure of complex systems can be visualized by reliability block diagrams, an example is given in Figure 3. Components are represented by boxes or nodes, and the system works when a path from the left end to the right exists which passes only through working components. In a system with n components, the state of the components can be expressed by the state vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$, with $x_i = 1$ if the ith component functions and $x_i = 0$ if not. The structure function $\phi : \{0, 1\}^n \to \{0, 1\}$, defined for all possible $\boldsymbol{x}$, takes the value 1 if the system functions and 0 if the system does not function for state vector $\boldsymbol{x}$ (Barlow and Proschan 1975). Most real-life systems are coherent, which means that $\phi(\boldsymbol{x})$ is non-decreasing in any of the components of $\boldsymbol{x}$, so system functioning cannot be improved by worse performance of one or more of its components. Furthermore, one can usually assume that $\phi(0, \dots, 0) = 0$ and $\phi(1, \dots, 1) = 1$.

The survival signature (Coolen and Coolen-Maturi 2012) is a summary of the structure function for systems with K groups of exchangeable components. Denoted by $\Phi(l_1, \dots, l_K)$, with $l_k = 0, 1, \dots, n_k$ for $k = 1, \dots, K$, it is defined as the probability for the event that the system functions given that precisely l_k of its n_k components of type k function, for each $k \in \{1, \dots, K\}$. Essentially, this creates a K-dimensional partition for the event $T_{\text{sys}} > t$, such that $R_{\text{sys}}(t) = P(T_{\text{sys}} > t)$ can be calculated using the law of total probability,

$$\begin{aligned} P(T_{\text{sys}} > t) &= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} P(T_{\text{sys}} > t \mid C_t^1 = l_1, \ldots, C_t^K = l_K) P\Big(\bigcap_{k=1}^{K} \{C_t^k = l_k\}\Big) \\ &= \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, \ldots, l_K) P\Big(\bigcap_{k=1}^{K} \{C_t^k = l_k\}\Big) \\ &= \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C_t^k = l_k), \end{aligned} \tag{12}$$

where $P(C_t^k = l_k)$ is the (predictive) probability that exactly l_k components of type k function at time t, and the last equality holds as we assume that components of different types are independent. Equation 12 shows that the survival signature allows to calculate the system reliability $P(T_{\text{sys}} > t)$ for arbitrary component failure time distributions, as the probabilities $P(C_t^k = l_k)$ can be determined from any failure time distribution. Note that for coherent systems, the survival signature $\Phi(l_1, \ldots, l_K)$ is non-decreasing in each l_k.

Posterior Predictive Distribution

In calculating the system reliability using (12), the component-specific predictive probabilities $P(C_t^k = l_k)$ need to use all information available at time t_{now}, which, in the Bayesian framework, are given by the posterior predictive distribution $P(C_t^k = l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k})$, $l_k = 0, 1, \ldots, n_k - e_k$. (Remember that e_k type k components have failed by t_{now}, such that there can be at most $n_k - e_k$ working components beyond time t_{now}.) This posterior predictive distribution is obtained as

$$\begin{aligned} &P(C_t^k = l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) \\ &\quad = \binom{n_k - e_k}{l_k} \int [P(T^k > t \mid T^k > t_{\text{now}}, \lambda_k)]^{l_k} \times \\ &\qquad\qquad [P(T^k \le t \mid T^k > t_{\text{now}}, \lambda_k)]^{n_k - e_k - l_k} f(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) \, \mathrm{d}\lambda_k \,. \end{aligned} \tag{13}$$

Now, by the Weibull assumption (2), one has

$$P(T^k \leq t \mid T^k > t_{\text{now}}, \lambda_k) = \frac{P(t_{\text{now}} < T^k \leq t \mid \lambda_k)}{P(T^k > t_{\text{now}} \mid \lambda_k)} = \frac{F(t \mid \lambda_k) - F(t_{\text{now}} \mid \lambda_k)}{1 - F(t_{\text{now}} \mid \lambda_k)} = 1 - e^{-\frac{t^{\beta_k} - (t_{\text{now}})^{\beta_k}}{\lambda_k}} . \tag{14}$$

With this and the posterior (6) substituted into (13), this gives

$$\begin{aligned} P(C_t^k = l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) &= \binom{n_k - e_k}{l_k} \int \left[e^{-\frac{t^{\beta_k} - (t_{\text{now}})^{\beta_k}}{\lambda_k}}\right]^{l_k} \left[1 - e^{-\frac{t^{\beta_k} - (t_{\text{now}})^{\beta_k}}{\lambda_k}}\right]^{n_k - e_k - l_k} \times \\ &\qquad \frac{(n_k^{(n)} y_k^{(n)})^{n_k^{(n)}+1}}{\Gamma(n_k^{(n)} + 1)} \lambda_k^{-(n_k^{(n)}+1)-1} e^{-\frac{n_k^{(n)} y_k^{(n)}}{\lambda_k}} \, \mathrm{d}\lambda_k \\ &= \binom{n_k - e_k}{l_k} \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \binom{n_k - e_k - l_k}{j} \frac{(n_k^{(n)} y_k^{(n)})^{n_k^{(n)}+1}}{\Gamma(n_k^{(n)} + 1)} \times \\ &\qquad \int \lambda_k^{-(n_k^{(n)}+1)-1} \exp\Big\{ - \frac{(l_k + j)(t^{\beta_k} - (t_{\text{now}})^{\beta_k}) + n_k^{(n)} y_k^{(n)}}{\lambda_k} \Big\} \, \mathrm{d}\lambda_k . \end{aligned} \tag{15}$$

The terms remaining under the integral form the core of an inverse gamma distribution (4) with parameters $n_k^{(n)} + 1$ and $n_k^{(n)} y_k^{(n)} + (l_k + j)(t^{\beta_k} - (t_{\text{now}})^{\beta_k})$, allowing to solve the integral using the corresponding normalization constant. We thus have, for $l_k \in \{0, 1, \ldots, n_k - e_k\}$,

$$
\begin{aligned}
&P(C_t^k = l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) \\
&= \binom{n_k - e_k}{l_k} \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \binom{n_k - e_k - l_k}{j} \left(\frac{n_k^{(n)} y_k^{(n)}}{n_k^{(n)} y_k^{(n)} + (l_k + j)\big(t^{\beta_k} - (t_\text{now})^{\beta_k}\big)} \right)^{n_k^{(n)}+1} \\
&= \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \frac{(n_k - e_k)!}{l_k! j! (n_k - e_k - l_k - j)!} \left(\frac{n_k^{(n)} y_k^{(n)}}{n_k^{(n)} y_k^{(n)} + (l_k + j)\big(t^{\beta_k} - (t_\text{now})^{\beta_k}\big)} \right)^{n_k^{(n)}+1} \\
&= \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \frac{(n_k - e_k)!}{l_k! j! (n_k - e_k - l_k - j)!} \times \\
&\qquad \left(\frac{n_k^{(0)} y_k^{(0)} + \sum_{i=1}^{e_k} (t_i^k)^{\beta_k} + (n_k - e_k)(t_\text{now})^{\beta_k}}{n_k^{(0)} y_k^{(0)} + \sum_{i=1}^{e_k} (t_i^k)^{\beta_k} + (n_k - e_k - l_k - j)(t_\text{now})^{\beta_k} + (l_k + j) t^{\beta_k}} \right)^{n_k^{(0)} + e_k + 1} .
\end{aligned}
\tag{16}
$$

These posterior predictive probabilities can also be expressed as a cumulative probability mass function (cmf)

$$
F(l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) = P(C_t^k \le l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) = \sum_{j=0}^{l_k} P(C_t^k = j \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}) . \tag{17}
$$

Optimizing over Sets of Parameters

Together with (16), (12) allows to calculate the system reliability $R_\text{sys}(t \mid t > t_\text{now})$ for fixed prior parameters $(n_k^{(0)}, y_k^{(0)})$, $k = 1, \ldots, K$. In Section 3, we argued for using sets of priors $\mathcal{M}^{(0)}$, which allow for vague and incomplete prior knowledge, and provide prior-data conflict sensitivity. We will thus use, for each component type, a set of priors $\mathcal{M}_k^{(0)}$ defined by varying $(n_k^{(0)}, y_k^{(0)})$ in a prior parameter set $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$, and the objective is to obtain the bounds

$$
\underline{R}_\text{sys}(t \mid t > t_\text{now}) = \min_{\Pi_1^{(0)}, \ldots, \Pi_K^{(0)}} R_\text{sys}\big(t \mid t > t_\text{now}, \cup_{k=1}^K \{\Pi_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}\}\big) , \tag{18}
$$

$$
\overline{R}_\text{sys}(t \mid t > t_\text{now}) = \max_{\Pi_1^{(0)}, \ldots, \Pi_K^{(0)}} R_\text{sys}\big(t \mid t > t_\text{now}, \cup_{k=1}^K \{\Pi_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}\}\big) , \tag{19}
$$

where we suppress in notation that $\underline{R}_{\text{sys}}(t \mid t > t_{\text{now}})$ and $\overline{R}_{\text{sys}}(t \mid t > t_{\text{now}})$ depend on prior parameter sets and data.

Equations (18) and (19) seem to suggest that a full $2K$-dimensional box-constraint optimization is necessary, but this is not the case. Remember that $\Phi(l_1, \ldots, l_k)$ from (12) is non-decreasing in each of its arguments $l_1, \ldots, l_K$, so if there is stochastic dominance in $F(l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k})$, then there is, for each component type k, a prior parameter pair in $\Pi_k^{(0)}$ that minimizes system reliability, and a prior parameter pair in $\Pi_k^{(0)}$ that maximizes system reliability, independently of the other component types. Indeed, stochastic dominance in $F(l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k})$ is provided for $y_k^{(0)}$. To see this, note that $y_k^{(0)}$ gives the mean expected lifetime for type k components. Thus, higher values for $y_k^{(0)}$ mean higher expected lifetimes for the components, which in turn increases the probability that many components survive until time t, and with it, decreases the propability of few or no components surviving, so in total giving low probability weight for low values of l_k, and high probability weight for high values of l_k. Therefore, for any fixed value of $n_k^{(0)}$, the lower bound of $F(l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k})$ for all l_k is obtained with $\overline{y}_k^{(0)}$, and the upper bound of $F(l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k})$ for all l_k is obtained with $\underline{y}_k^{(0)}$. There is however no corresponding result for $n_k^{(0)}$, such that different values of $n_k^{(0)}$ may minimize (or maximize) $F(l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k})$ at different l_k's. Therefore, the $n_k^{(0)}$ values for lower and upper system reliability bounds are obtained by numeric optimization.

Writing out (12), one obtains

$$\begin{aligned} \underline{R}_{\text{sys}}(t \mid t > t_{\text{now}}) &= \min_{\Pi_1^{(0)},\ldots,\Pi_K^{(0)}} R_{\text{sys}}\big(t \mid t > t_{\text{now}}, \cup_{k=1}^K \{\Pi_k^{(0)}, \boldsymbol{t}^k_{e_k;n_k}\}\big) \\ &= \min_{\substack{n_1^{(0)} \in [\underline{n}_1^{(0)}, \overline{n}_1^{(0)}] \\ \vdots \\ n_K^{(0)} \in [\underline{n}_K^{(0)}, \overline{n}_K^{(0)}]}} \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1, \ldots, l_K) \prod_{k=1}^K P(C_t^k = l_k \mid n_k^{(0)}, \underline{y}_k^{(0)}, \mathbf{t}^k_{e_k;n_k}), \end{aligned} \tag{20}$$

such that a K-dimensional box-constraint optimization is needed to obtain $\underline{R}_{\text{sys}}(t \mid t > t_{\text{now}})$. The result for $\overline{R}_{\text{sys}}(t \mid t > t_{\text{now}})$ is completely analogous. Computing time can furthermore be saved by computing only those summation terms for which $\Phi(l_1, \dots, l_K) > 0$.

We have implemented the method in the statistical computing environment `R` (R Core Team 2017), using box-constraint optimization via option `L-BFGS-B` of `optim`. Code for reproducing all results and figures for the examples in Section 6 below is available upon request.

ELICITATION OF PRIOR PARAMETER SETS

To represent expert knowledge on component failure times through bounds for $y_k^{(0)}$ and $n_k^{(0)}$, one can refer to the interpretations as given in Section 2: $y_k^{(0)}$ is the prior expected value of λ_k, where λ_k is linked to expected component lifetimes through (3). $n_k^{(0)}$ can be seen as pseudocount, indicating how strong expert knowledge is trusted in comparison to a sample of size n. Crucially, the approach allows the expert to give ranges $[\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ and $[\underline{n}_k^{(0)}, \overline{n}_k^{(0)}]$ instead of requiring a precise answer.

It is also possible to directly link $n_k^{(0)}$ and $y_k^{(0)}$ to observed lifetimes using a prior predictive distribution. Dropping the component index k for ease of notation, this is given by

$$
\begin{aligned}
f(t \mid n^{(0)}, y^{(0)}) &= \int f(t \mid \lambda) f(\lambda \mid n^{(0)}, y^{(0)}) \, \mathrm{d}\lambda \\
&= \beta \, t^{\beta - 1} \, (n^{(0)} + 1) \frac{(n^{(0)} y^{(0)})^{n^{(0)}+1}}{(n^{(0)} y^{(0)} + t^{\beta})^{n^{(0)}+2}} \,. \qquad (21)
\end{aligned}
$$

Replacing the prior parameters $n^{(0)}$ and $y^{(0)}$ with their posterior counterparts $n^{(n)}$ and $y^{(n)}$ as defined in (5), the effect of virtual observations on (21), or the corresponding reliability function, can be determined. This allows to determine $n^{(0)}$ and $y^{(0)}$ through a number of 'what-if' scenarios, by asking the expert to state what (s)he would expect to learn from observing certain virtual data.

This strategy is known as pre-posterior analysis, being first advocated by Good (1965, p. 19). We recommend to check whether the effects of $\Pi_k^{(0)}$ on the inference of interest (this may not always be the full reliability function) reasonably reflect an expert's beliefs before the data and in case some specific data become available, both data agreeing with initial beliefs and surprising data. Essentially, we advise to do an analysis like in our examples in Section 6 below, using hypothetical data.

To elicit a meaningful prior distribution, or a set of prior distributions, it is important to ask questions which enable experts to stay close to their actual expertise. Coolen (1996) discussed the possibility of generalizing the usual conjugate prior distributions, for parameters of exponential family models, by including pseudo-data which are right-censored. If the real data set contains such values, then such generalized priors do not lead to more computational complexities, while they can have several advantages. In addition to providing slightly more general classes of prior distributions through an additional hyperparameter, they may enable more realistic elicitation of expert judgements, for example if the expert has no experience with certain components past a specific life time. For more details we refer to Coolen (1996), it should be noted that adopting such generalized prior distributions may also provide more flexibility for modelling the effects of prior-data conflict, this is left as a topic for future research.

EXAMPLES

As illustrative example, consider a simplified automotive brake system with four types of component. The master brake cylinder (M) activates all four wheel brake cylinders (C1 – C4), which in turn actuate a braking pad assembly each (P1 – P4). The hand brake mechanism (H) goes directly to the brake pad assemblies P3 and P4; the car brakes when at least one brake pad assembly is actuated. The system layout is depicted in Figure 3, with those components marked that we assume to fail in each of the three cases studied below. The values for $\Phi \notin \{0, 1\}$ for the complete system are given in Table 1.

A fixed prior setting will be combined with three different data scenarios, where one observes failure times in accordance with prior expectations in the first case, surprisingly early failures in the second case, and surprisingly late failures in the third case. In each case, it is assumed that C2, C3, P2 and P3 fail, only the failure times are varied, investigating the effect of learning about these failures on the component level. We then discuss effects on posterior reliability bounds for the running system for all three cases.

Prior Assumptions

The prior assumptions, which one can imagine to be determined by an expert, or by a combination of expert knowledge and component test data, are given by the prior parameter sets $\Pi_k^{(0)}$, $k = \mathrm{M}, \mathrm{H}, \mathrm{C}, \mathrm{P}$, as described in Table 2. There, $\underline{\mathrm{E}}[T_i^k]$ and $\overline{\mathrm{E}}[T_i^k]$ give the lower and upper bound for expected component lifetimes, respectively, which then have been transformed to bounds for the scale parameter using (3), resulting in $\underline{y}_k^{(0)}$ and $\overline{y}_k^{(0)}$. For example, according to the expert, the mean time to failure for component type M is between 5 and 8 time units, leading to $\underline{y}_{\mathrm{M}}^{(0)} = 75.4$ and $\overline{y}_{\mathrm{M}}^{(0)} = 244.1$, and the expert considers his knowledge on these expected lifetime bounds as having the strength of at least 2 and at most 5 observations.

These prior assumptions for the four component types are visualized in Figure 4, showing the sets of reliability functions corresponding to the prior predictive density (21). The figure thus displays the bounds for the probability that a single component, having been put under risk at time 0, will have failed by time t. The top left graph in Figure 8 shows what the prior assumptions on components signify for the system, depicting the prior bounds for the system reliability on a scale of time elapsed since system startup. For example, the prior probability of the system to survive until time 10 is between 0.03% and 6.91%.

Case 1: Failure Times as Expected

In the first case, we observe $t_1^{\mathrm{C}} = 6$, $t_2^{\mathrm{C}} = 7$, $t_1^{\mathrm{P}} = 3$, $t_2^{\mathrm{P}} = 4$, and observe the running system until $t_{\mathrm{now}} = 8$, i.e., $t_1^{\mathrm{M}} = t_1^{\mathrm{H}} = t_3^{\mathrm{C}} = t_4^{\mathrm{C}} = t_3^{\mathrm{P}} = t_4^{\mathrm{P}} = 8^+$. (Note that component failure times are numbered by order, not by component number in the system layout.) These observations correspond more or less to prior expectations, and the corresponding posterior predictive component distributions are given in Figure 5. In analogue to Figure 4, Figure 5 displays the bounds for the probability that a single component, having been put under risk at time 0, will have failed by time t, after having seen these (partly censored) observations. For easy comparisons, Figure 5 also contains the prior bounds from Figure 4.

We see that the graphs for M and H do not change dramatically, as there is only one component of each in the system to learn from. For C, the bounds have considerably narrowed, showing the effect of having seen the four observations $t_1^{\mathrm{C}} = 6$, $t_2^{\mathrm{C}} = 7$, $t_3^{\mathrm{C}} = t_4^{\mathrm{C}} = 8^+$. The bounds for P have not narrowed as much; this is due to the two right-censored observations $t_3^{\mathrm{P}} = t_4^{\mathrm{P}} = 8^+$; from the viewpoint of the prior, surviving past time 8 is already quite unusual.

Case 2: Surprisingly Early Failure Times

For the second case, with $t_1^{\mathrm{C}} = 1$, $t_2^{\mathrm{C}} = 2$, $t_1^{\mathrm{P}} = 0.25$, $t_2^{\mathrm{P}} = 0.5$ and $t_{\mathrm{now}} = 2$ (so $t_1^{\mathrm{M}} = t_1^{\mathrm{H}} = t_3^{\mathrm{C}} = t_4^{\mathrm{C}} = t_3^{\mathrm{P}} = t_4^{\mathrm{P}} = 2^+$), we assume to observe surprisingly early failures; the corresponding posterior predictive component distributions are given in Figure 6. Having observed the system only until $t = 2$, the data are not very informative, such that prior and posterior predictive reliability bounds are very similar. For C, the effect of the early failures is however still visible, and posterior imprecision, i.e., the range between lower and upper posterior bound, is notably larger as compared to prior imprecision, and substantially larger than posterior imprecision in case 1. The effect for P is less pronounced, mainly because observing $\boldsymbol{t}^{\mathrm{P}} = (0.25, 0.5, 2^+, 2^+)$ for P is less extreme as observing $\boldsymbol{t}^{\mathrm{C}} = (1, 2, 2^+, 2^+)$ for C.

Case 3: Surprisingly Late Failure Times

For the third case, we assume to observe surprisingly late failures, namely $t_1^{\mathrm{C}} = 11$, $t_2^{\mathrm{C}} = 12$, $t_1^{\mathrm{P}} = 8$, $t_2^{\mathrm{P}} = 9$, and $t_{\mathrm{now}} = 12$ (so $t_1^{\mathrm{M}} = t_1^{\mathrm{H}} = t_3^{\mathrm{C}} = t_4^{\mathrm{C}} = t_3^{\mathrm{P}} = t_4^{\mathrm{P}} = 12^+$); the corresponding posterior predictive component distributions are given in Figure 7. Having observed the system for a much longer time than in case 2, the data contain now much more information, resulting in considerable differences between prior and posterior bounds. The effect of these surprisingly late failures is most prominent for P, with a set considerably shifted to the right and having very wide posterior bounds. The posterior set for C also indicates that, after having seen these late failures, one expects type C components to fail much later. This effect is also visible for M and H, but is weaker for them as there is only one component of each in the system.

Reliability Bounds for the Running System

Figure 8 depicts the set of prior system reliability functions, together with the sets of posterior system reliability function for the three cases, on a scale of elapsed time since system startup. Due to this time scale, posterior system reliability is 1 at t_{now}, as it is known that the system has survived until $t_{\text{now}} = 8, 2, 12$ in case 1,2,3, respectively. For all three cases, the posterior bounds drop faster after t_{now} than the prior bounds drop after $t = 0$ since the components in the system have aged until t_{now} and so are expected to fail sooner. In the case of surprisingly early failures, posterior bounds are mostly within prior bounds, this is due to t_{now} being close to 0 and weakly informative data in this scenario; the posterior bounds are nevertheless wider than those for case 1; posterior bounds are widest for case 3. This most visible in Figure 9, which shows $\overline{R}_{\text{sys}}(t) - \underline{R}_{\text{sys}}(t)$, the difference between upper and lower bound of prior and posterior system reliability. The left panel shows imprecision on the scale of elapsed time since system startup like in Figure 8; the right panel shows imprecision on the scale of prospective time instead, indicating how far in the future periods are for which estimation of system reliability is most uncertain. Posterior imprecision is indeed considerably lower in case 1, where failure times were more or less like expected. On the prospective timescale, one can see that periods of heightened uncertainty are closer to the present for the posteriors, while uncertainty is considerably reduced for periods further in the future.

CONCLUDING REMARKS

In this paper we presented a robust Bayesian approach to reliability estimation for systems of arbitrary layout, and showed how the use of sets of prior distributions results in increased imprecision, i.e., more cautious probability statements, in case of prior-data conflict (cases 2 and 3 in Section 6), while giving more precise reliability bounds when prior and data are in agreement (case 1 in Section 6). The parameters through which prior information is encoded have a clear interpretation and are thus easily elicited, either directly or with help of the prior predictive (21). Calculation of lower and upper predictive system reliability bounds is tractable, requiring only a simple K-dimensional box-constrained optimization in Equation (20).

We think that increased imprecision is an appropriate tool for mirroring prior-data conflict when considering sets of priors as is done in both the robust Bayesian and imprecise probability framework. We want to emphazise, however, that this tool may be useful already for just highlighting 'conflict' between multiple information sources, and that we do not think that the resulting set of posteriors, although it can form a meaningful basis, must necessarily be used for all consequential inferences, as a strict Bayesian would posit. We believe an analyst is free to reconsider any aspect of a model (of which the choice of prior can be seen to form a part) after seeing the data, and so may use our method only for becoming aware of a conflict between prior and data.

The employed robust Bayesian setting provides many further modelling opportunities beyond the explicit reaction to prior-data conflict. These opportunities have not yet been explored and provide a wide field for further research. An example is our currently ongoing investigation into extending the present model to allow also for an appropriate reflection of very strong agreement between prior and data (Walter and Coolen 2016). Another interesting avenue for future research may be to investigate measures of prior-data conflict based on imprecise probabilistic inference. These measures could be based on changes in imprecision for certain key probabilities, like the system reliability at a prescribed mission time.

There are many aspects to further develop in our analysis and modeling. The general approaches used in this paper, namely the use of sets of conjugate priors for component lifetime models and the survival signature to calculate the system reliability, can be used with other parametric component lifetime distributions that form a canonical exponential family, since for such distributions, a canonical conjugate prior using the same canonical parameters $n^{(0)}$ and $y^{(0)}$ can be constructed, for details see, e.g., Bernardo and Smith (2000, p. 202 and 272f), or Walter (2013, p. 8). Likewise straightforward to implement is, e.g., the analysis of the effect of replacing failed components in the system on system reliability bounds. Criteria for the trade-off between the cost of replacement and the gain in reliability would have to be adapted for the interval output of our model, leading to very interesting research questions.

To estimate the shape parameters β_k together with the scale parameters λ_k, one could follow standard Bayesian approaches and use a finite discrete distribution for β_k. Developing this together with suitable sets of priors for λ_k, in particular to show the effect of prior-data conflict, is another interesting challenge for future research.

Another further important aspect in system reliability we have not accounted for yet is the possibility of common-cause failures, i.e., failure events where several components fail at the same time due to a shared or common root cause. This could be done by combining the common-cause failure model approaches of Troffaes et al. (2014) and Coolen and Coolen-Maturi (2015).

On a more abstract level, the choice for the set of prior parameters (generating the set of prior component failure distributions) as $[\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ has the advantage of allowing for easy elicitation and tractable inferences, but it may not be suitable to reflect certain kinds of prior knowledge. Also, as studied in (Walter et al. 2011) and (Walter 2013, §3.1), the shape of the prior parameter set has a crucial influence on model behaviour like the severity of prior-data conflict reaction. As noted in (Walter 2013, pp. 66f), more general prior parameter set shapes are possible in principle, but may be more difficult to elicit and make calculations more complex.

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List of Tables

1 Survival signature values $\notin \{0, 1\}$ for the simplified automotive brake system depicted in Figure 3. 24

2 Prior parameter sets for the four component types. 25

Table 1. Survival signature values $\notin \{0, 1\}$ for the simplified automotive brake system depicted in Figure 3.

M	H	C	P	Φ	M	H	C	P	Φ
1	0	1	1	0.25	1	0	2	1	0.50
1	0	1	2	0.50	1	0	2	2	0.83
1	0	1	3	0.75	1	0	3	1	0.75
0	1	0	1	0.50	1	1	0	1	0.50
0	1	0	2	0.83	1	1	0	2	0.83
0	1	1	1	0.50	1	1	1	1	0.62
0	1	1	2	0.83	1	1	1	2	0.92
0	1	2	1	0.50	1	1	2	1	0.75
0	1	2	2	0.83	1	1	2	2	0.97
0	1	3	1	0.50	1	1	3	1	0.88
0	1	3	2	0.83					
0	1	4	1	0.50					
0	1	4	2	0.83					

Table 2. Prior parameter sets for the four component types.

k	β_k	$\underline{\mathrm{E}}[T_i^k]$	$\overline{\mathrm{E}}[T_i^k]$	$\underline{y}_k^{(0)}$	$\overline{y}_k^{(0)}$	$\underline{n}_k^{(0)}$	$\overline{n}_k^{(0)}$
M	2.5	5	8	75.4	244.1	2	5
H	1.2	2	20	2.5	39.2	1	10
C	2	8	10	81.5	127.3	1	5
P	1.5	3	4	6.1	9.3	1	10

List of Figures

1 Prior and posterior cdf for λ given surprising observations; the conflict between prior assumptions and data is averaged out, with a more pointed posterior giving a false sense of certainty. 27

2 Set of prior and posterior cdfs for λ for two surprising observations $t_1 = 1, t_2 = 2$ (left) and two unsurprising observations $t_1 = 10, t_2 = 11$ (right). 28

3 Reliability block diagram for a simplified automotive brake system, with those components marked that we assume to fail in the three scenarios. 29

4 Sets of prior predictive reliability functions for the four component types, illustrating the choice of prior parameter sets $\Pi_k^{(0)}$, $k = \mathrm{M, H, C, P}$. 30

5 Sets of posterior predictive reliability functions for the four component types for observations in line with prior expectations (case 1). 31

6 Sets of posterior predictive reliability functions for the four component types for surprisingly early failures (case 2). 32

7 Sets of posterior predictive reliability functions for the four component types for surprisingly late failures (case 3). 33

8 Sets of prior and posterior system reliability functions for the three cases on a time showing time elapsed since system startup. 34

9 Imprecision of prior and posterior system reliability sets. 35

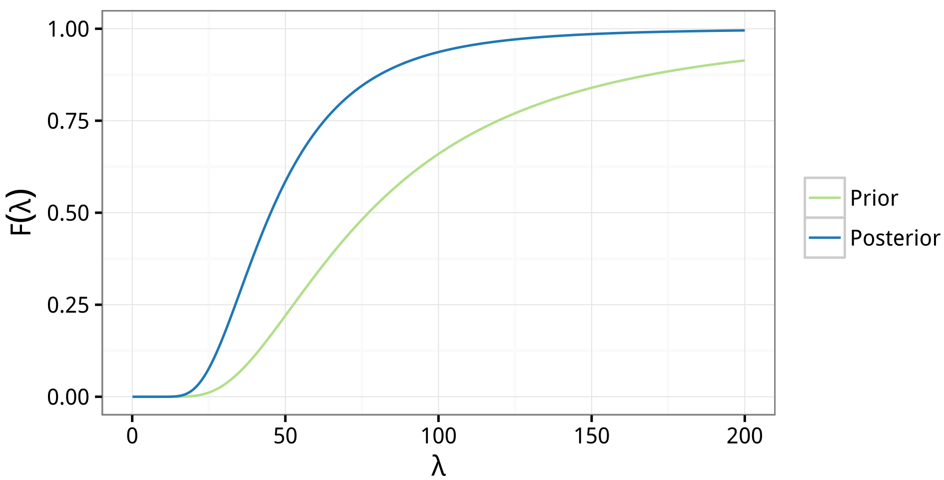

Fig. 1. Prior and posterior cdf for λ given surprising observations; the conflict between prior assumptions and data is averaged out, with a more pointed posterior giving a false sense of certainty.

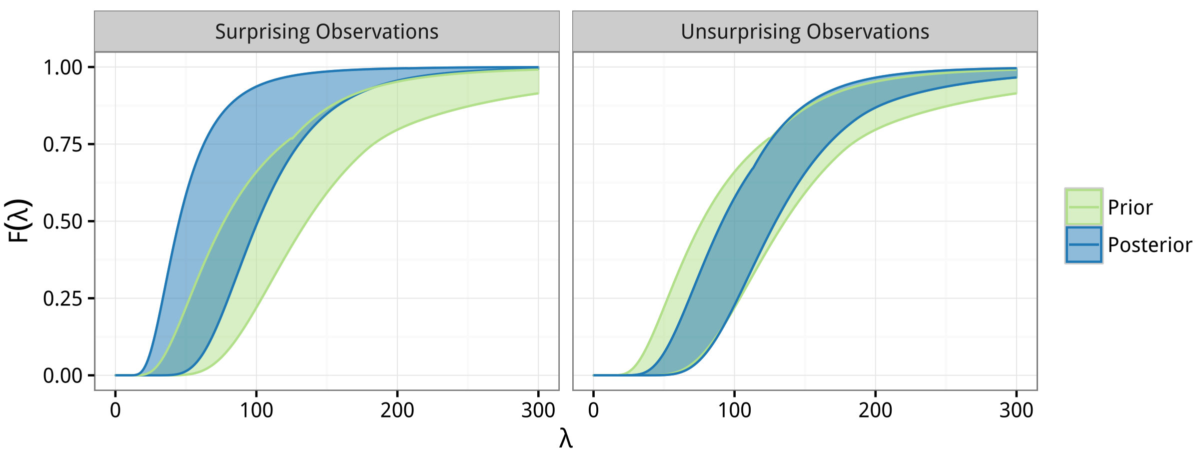

Fig. 2. Set of prior and posterior cdfs for λ for two surprising observations $t_1 = 1$, $t_2 = 2$ (left) and two unsurprising observations $t_1 = 10$, $t_2 = 11$ (right).

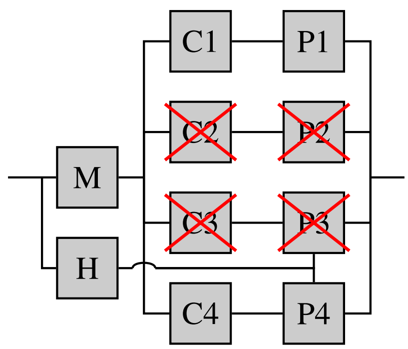

Fig. 3. Reliability block diagram for a simplified automotive brake system, with those components marked that we assume to fail in the three scenarios.

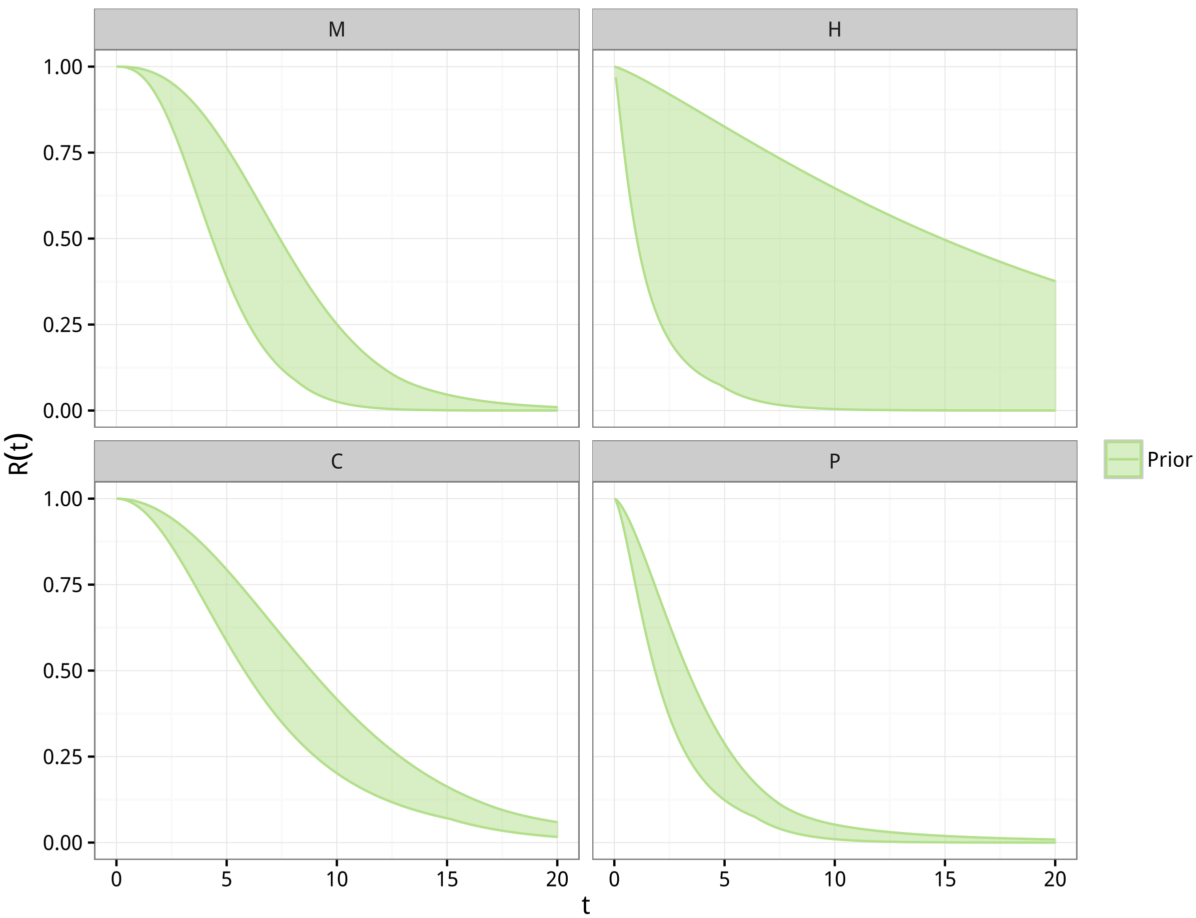

Fig. 4. Sets of prior predictive reliability functions for the four component types, illustrating the choice of prior parameter sets $\Pi_k^{(0)}$, $k = \text{M}, \text{H}, \text{C}, \text{P}$.

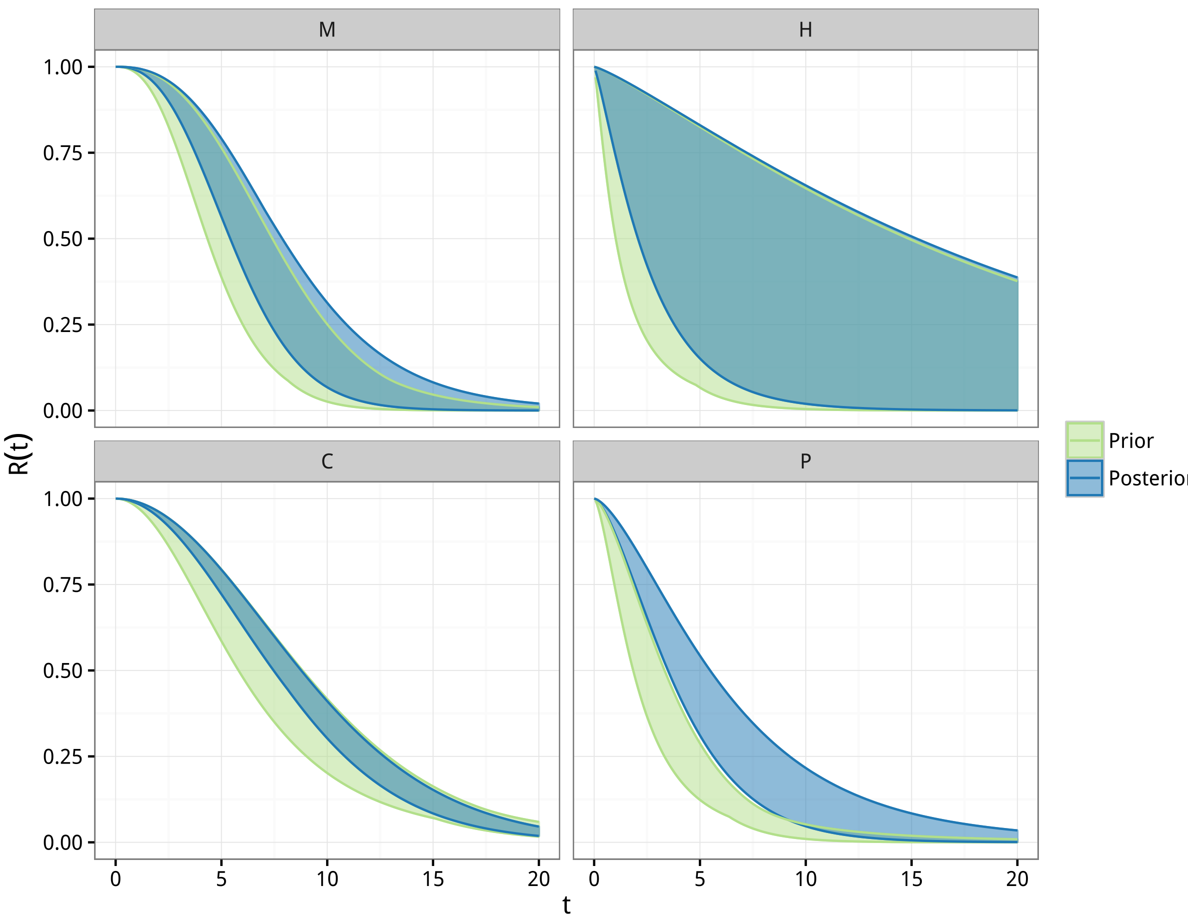

Fig. 5. Sets of posterior predictive reliability functions for the four component types for observations in line with prior expectations (case 1).

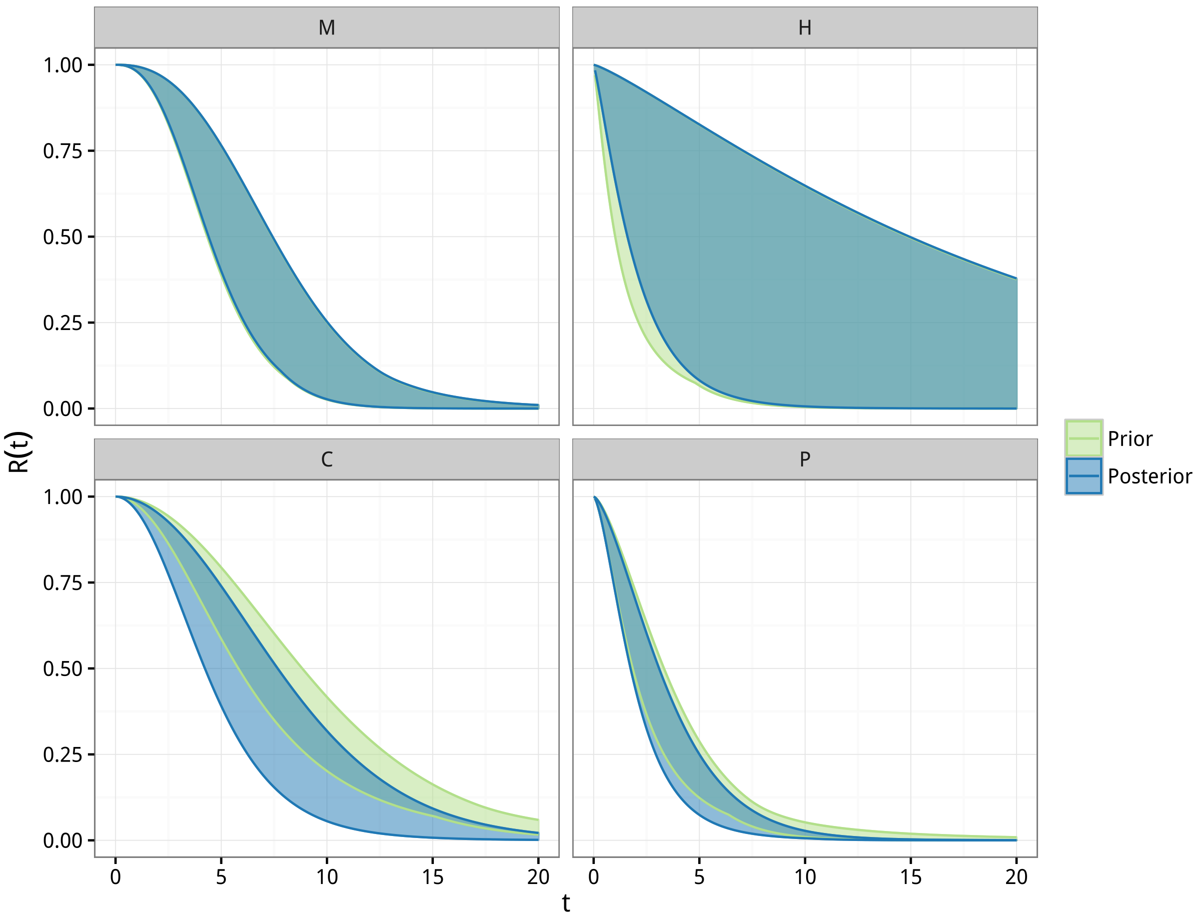

Fig. 6. Sets of posterior predictive reliability functions for the four component types for surprisingly early failures (case 2).

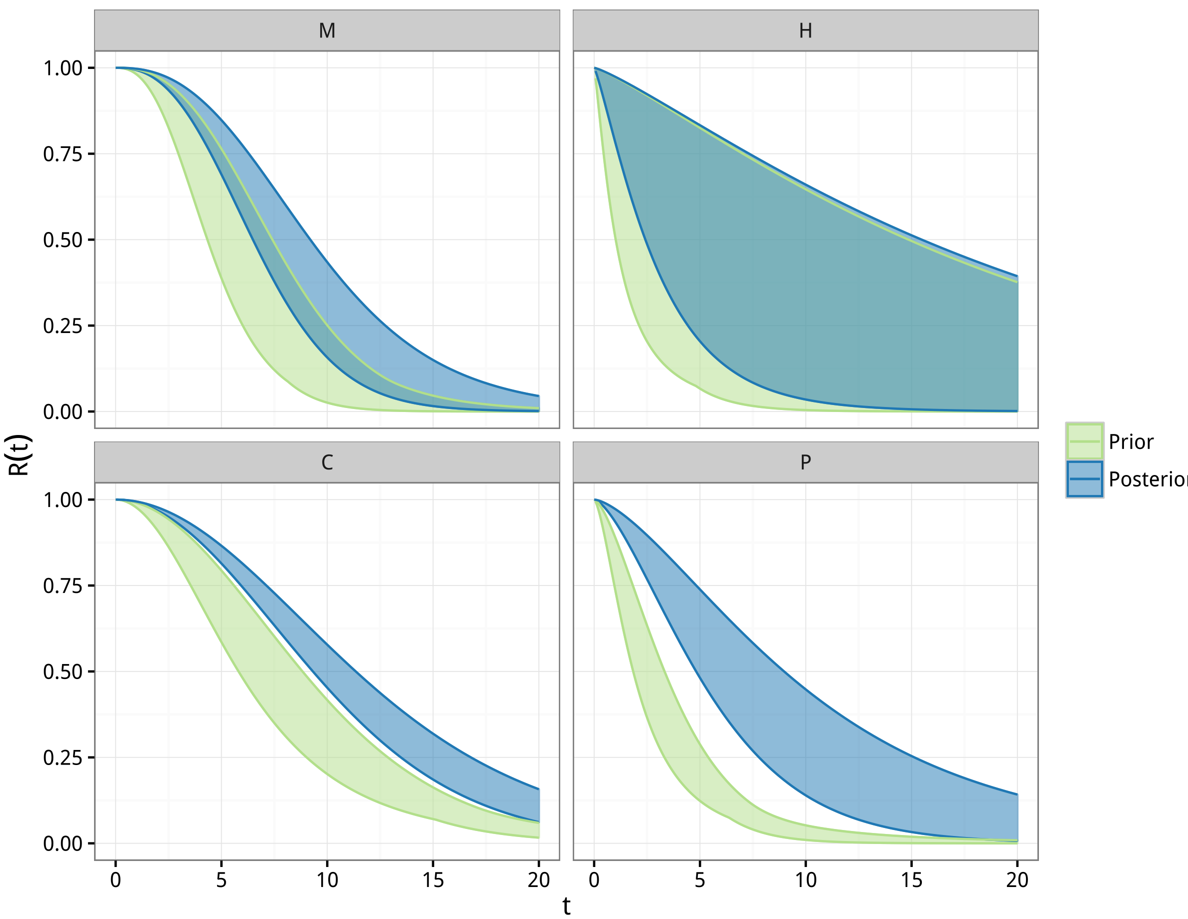

Fig. 7. Sets of posterior predictive reliability functions for the four component types for surprisingly late failures (case 3).

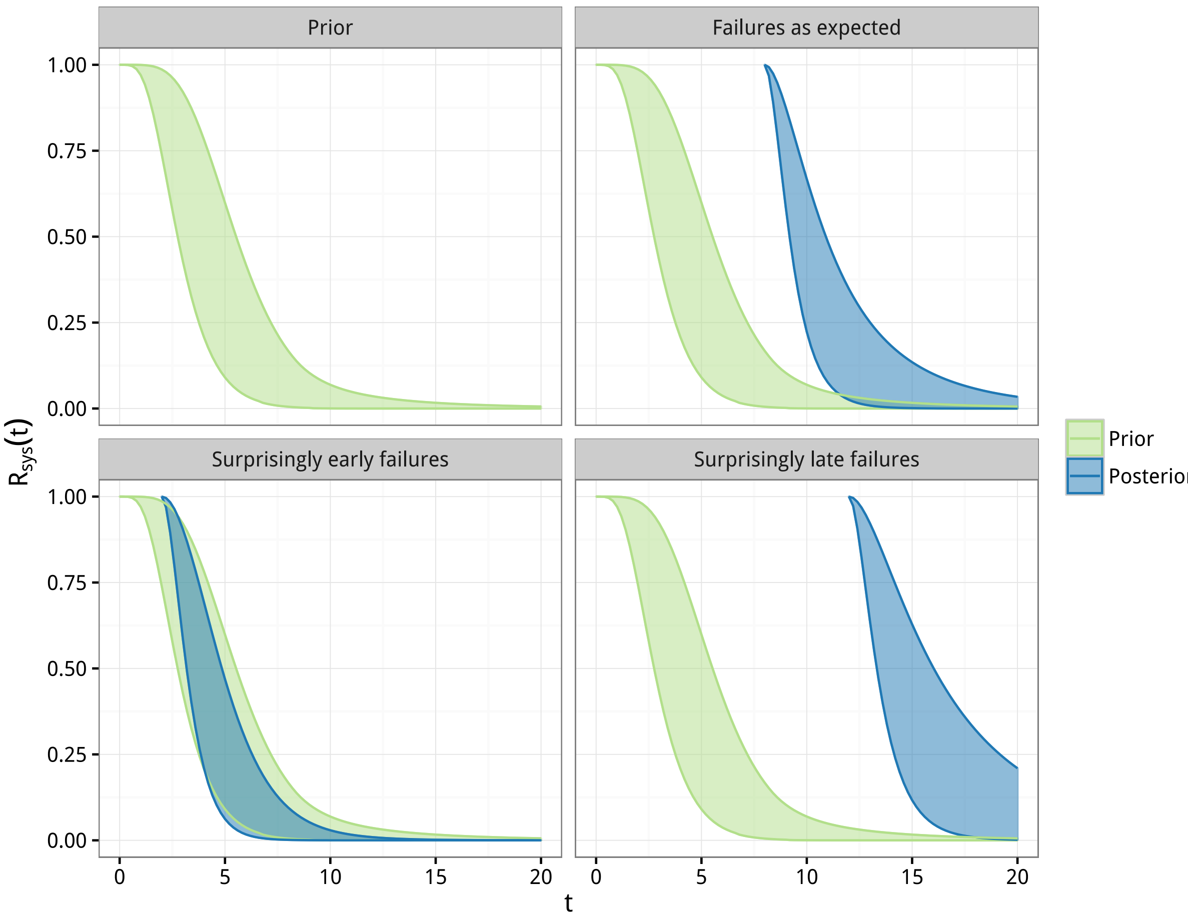

Fig. 8. Sets of prior and posterior system reliability functions for the three cases on a time showing time elapsed since system startup.

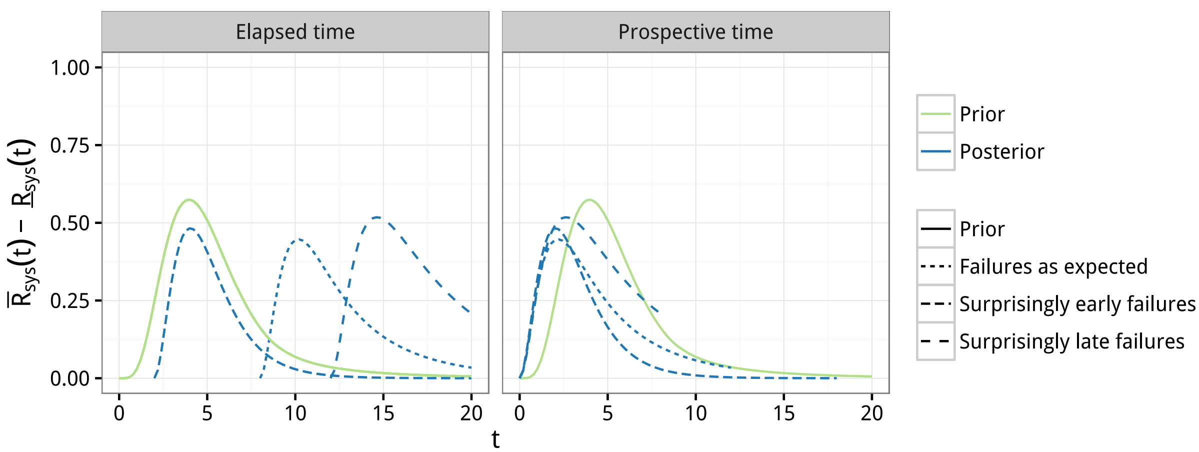

Fig. 9. Imprecision of prior and posterior system reliability sets.